Announcements

1) HW 1 due today, 5:00

2) HW 2 up either tonight or tomorrow

Recall

Theorem: (unitary diagonalization)

If $A \in \mathbb{C}^{m\times m}$ satisfies $A = A^*$, then A is unitarily diagonalizable; i.e., $\exists$ diagonal matrix $D \in \mathbb{C}^{m\times m}$ and Q unitary, $Q \in \mathbb{C}^{m\times m}$, with

$$\boxed{A = QDQ^*}$$

Consequence: A^*A (for $A \in \mathbb{C}^{m \times n}$) is always equal to its adjoint. Moreover, if λ is an eigenvalue for A^*A and v is an associated eigenvector, then

$$\boxed{(A^*Av)^*v}$$

$$= (\lambda v)^* v$$

$$= \bar{\lambda}\, v^* v$$

v^*v is a positive scalar.

However, by adjoint properties,

$$\boxed{(A^*Av)^* v}$$

$$= (Av)^*(Av) \quad \text{(adjoint property)}$$

$$\geq 0.$$

If zero, then

$$\lambda v^* v = 0 \Rightarrow \lambda = 0.$$

If bigger than zero,

$$\bar{\lambda} v^* v > 0,$$

$$\bar{\lambda} > 0.$$

This shows that the eigenvalues of A^*A are non-negative real numbers.

We can write

$$A^*A = QDQ^*$$

with the diagonal entries of D nonnegative real numbers.

Definition: (absolute value)

Let $A \in \mathbb{C}^{m \times n}$. We define the absolute value of A to be the matrix

$$\boxed{|A| = Q\sqrt{D}\,Q^*}$$

where

$$A^*A = QDQ^*$$

and $\sqrt{D}$ is the diagonal matrix whose entries are the entries of D, square-rooted.

Fact about complex numbers

$z \in \mathbb{C}$.

Can write

$$z = e^{i\theta}|z| \quad \text{(polar decomposition)}$$

for some θ, $0 \leq \theta < 2\pi$.

For complex numbers,

$$|e^{i\theta}| = 1 \quad \forall\, \theta \in \mathbb{R}.$$

Example 1:

Let $z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$.

Then $|z| = 5\left|\frac{\sqrt{3}}{2} + \frac{i}{2}\right|$

$= 5$.

$$\theta = \frac{\pi}{6} = \arctan\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

We write

$$z = 5e^{\frac{\pi i}{6}}.$$

Theorem: (polar decomposition)

$A \in \mathbb{C}^{m \times n}$. Then there exists a matrix $Q \in \mathbb{C}^{m \times n}$ with orthonormal columns such that

$$\boxed{A = Q|A|}$$

(polar decomposition for matrices)

Singular value decomposition

(reduced)

Given $A \in \mathbb{C}^{m \times n}$, write

$A = Q|A|$ in polar

decomposition. Diagonalize

$|A| = R\sqrt{D}R^*$.

Let $P \in \mathbb{C}^{n\times n}$ be the permutation unitary that rearranges the entries on the diagonal of $\sqrt{D}$ so that in the (diagonal) matrix

$$P\sqrt{D}\,P^* = \hat{\Sigma},$$

$$(\hat{\Sigma})_{1,1} \geq (\hat{\Sigma})_{2,2} \geq \cdots \geq (\hat{\Sigma})_{n,n}$$

We write

$$\hat{U} = QRP^* \in \mathbb{C}^{m \times n}$$

(orthonormal columns)

and

$$\hat{V} = RP^* \in \mathbb{C}^{n \times n}$$

(unitary).

Then $\boxed{A = \hat{U}\hat{\Sigma}(\hat{V})^*}$ the reduced singular value decomposition of A.

Facts

1) Every matrix has a singular value decomposition, unlike eigenvalue decompositions.

2) A real matrix has a real singular value decomposition: $\hat{U}$ and $\hat{V}$ are real matrices.

Matlab Calling Command

svd(A) gives the diagonal entries of $\hat{\Sigma}$.

These are called the singular values of A.

zero

$$[U, D, V] = \text{svd}(A, 0)$$

gives the reduced singular value decomposition of A, with $D = \hat{\Sigma}$.

Example 2 :

$$A = \begin{bmatrix} 1 & 3 \\ 5 & i \\ -6i & 10 \end{bmatrix}$$

Singular value decomposition of A (4 decimal places)

$$\hat{U} = \begin{bmatrix} -.0550 + .1972i & .1817 + .42i \\ -.2951 - .003i & .8654 + .0065i \\ -.0303 + .9328i & -.0646 + .1936i \end{bmatrix}$$

$$\hat{\Sigma} = \begin{bmatrix} 12.4641 & 0 \\ 0 & 4.0799 \end{bmatrix}$$

$$\hat{V} = \begin{bmatrix} -.5718 & .8204 \\ -.0378+.8195i & -.0264+.5712i \end{bmatrix}$$

Full SVD

Start with the reduced SVD

$$A = \hat{U}\hat{\Sigma}(\hat{V})^*.$$

Let Σ be the $m \times n$ diagonal matrix with rows 1 through n the rows of $\hat{\Sigma}$ and rows $n+1$ to m all zero.

Let U be the $m \times m$ unitary matrix obtained from $\hat{U}$ by augmenting by $(m-n)$ orthonormal m-vectors which are orthogonal to the columns of $\hat{U}$ (Gram-Scmidt process)

$$V = \hat{V}.$$

Then

$\boxed{A = U \Sigma V}$ is the

full singular value decomposition of A.

Matlab: [U,D,V] = svd(A)